

# Study of some semi-linear elliptic equation

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## Abstract

We propose in this paper to study nodal solutions of some nonlinear elliptic equations derived from the famous equation of Brezis-Nirenberg and we analyse in some cases the possible singularities of radial solutions at the origin.

**Key Words and Phrases:** elliptic equations, nodal solutions, Sturm comparison theorem, Brezis-Nirenberg equation.

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## 1 Main Results

In this paper we focus on the study of the equation

$$\Delta u + u - |u|^{-2\theta}u = 0, \text{ in } \mathbb{R}^d, \quad (1)$$

with  $d > 1$ , and  $0 < \theta < \frac{1}{2}$ . We focus essentially on radial solutions. The radial version of problem (1) provided with the value of the solution  $u$  at the origin is

$$\begin{cases} u'' + \frac{d-1}{r}u' + u - |u|^{-2\theta}u = 0 & , \quad r \in (0, +\infty), \\ u(0) = a & , \quad u'(0) = 0, \end{cases} \quad (2)$$

where  $a \in \mathbb{R}$ . In the rest of the whole paper we denote

$$g(s) = 1 - |s|^{-2\theta}, \quad f(s) = sg(s) = s - s|s|^{-2\theta} \text{ and } F(s) = \frac{s^2}{2} \left(1 - \frac{|s|^{-2\theta}}{1-\theta}\right).$$

We propose to recall some results already known or easy to handle on the study of problem (1) or one of the problems related and cited above. To do this we recall the properties of  $g$ ,  $f$  and  $F$ . Because of the parity properties of these

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functions, we only provide their variations on  $(0, +\infty)$ .

$s$	0	1	$+\infty$
$g'(s)$	$\parallel$	+	+
$g(s)$	$-\infty$	$\emptyset$	1
$g(s)$	0	-	0

$s$	0	$s_\theta$	1	$+\infty$
$f'(s)$	$\parallel$	-	$\emptyset$	+
$f(s)$	$\emptyset$	$f(s_\theta)$	$\emptyset$	$+\infty$
$f(s)$	0	-	0	+

$s$	0	1	$p$	$+\infty$
$F'(s)$	$\emptyset$	-	$\emptyset$	+
$F(s)$	$\emptyset$	$F(1)$	$\emptyset$	$+\infty$
$F(s)$	0	-	0	+

The parameter  $p = \frac{1}{(1-\theta)^{\frac{1}{\theta}}}$  is the unique real number in  $(1, +\infty)$  such that  $F(p) = 0$ . We recall finally that we shall use many times the energy of the solution  $u$  defined for  $r \geq 0$  by

$$E(r) = \frac{1}{2}u'^2(r) + \int_0^{u(r)} sg(s)ds = \frac{1}{2}u'^2(r) + \int_0^{u(r)} f(s)ds = \frac{1}{2}u'^2(r) + F(u(r)).$$

The first result is stated as follows.

**Theorem 1.1** *The solution  $u$  of problem (2) is oscillating around 1 or -1 for any  $a \in ]-1, 1[ \setminus \{0\}$  with no zeros in  $(0, \infty)$ .*

Next we study the case where the origin value  $u(0) = a$  is not in the  $\pm 1$ -attractive zone. We prove that there are also different zones to be distinguished. We obtained the following result.

**Theorem 1.2** *i. For  $1 < a < p$ , the solution  $u$  of problem (2) .....*

ii. For  $a > p$ , the solution  $u$  of problem (2) .....

The following result deals with the existence and uniqueness of the solution.

**Corollary 1.1** *For all  $a \in ]1, p[$ , problem (2) has a unique solution  $u > 0$  which is oscillatory around 1.*

## 2 On the existence and uniqueness of solutions

**Lemma 2.1** *For all  $a \in (0, p)$ , the solution  $u$  of (2) satisfies the assertion*

$$u(\zeta) = 0, \text{ for some } \zeta \implies u'(\zeta) \neq 0,$$

*except if  $u \sim 0$ .*

**Lemma 2.2** *For all  $a \in ]0, p[$ , with  $p = \frac{1}{(1-\theta)^{\frac{1}{2\theta}}}$ , problem (2) has a unique positive solution  $u$ .*

Indeed, denote for  $r \in (0, +\infty)$  and consider the system

$$\begin{cases} u(r) = a + \int_0^r v(s)ds, \\ v(r) = -\frac{1}{r^{d-1}} \int_0^r s^{d-1} u(s)g(u(s))ds. \end{cases} \quad (3)$$

Using standard arguments from iterative methods in functional analysis, we observe that such a system has a unique local solution  $(u, v)$  on  $r \in (0, \delta)$  for  $\delta > 0$  small enough. The solution satisfies  $u(0) = a$ ,  $v(0) = 0$ . Furthermore,  $u > 0$ ,  $v < 0$ , and  $u$  and  $v$  are  $\mathcal{C}^2$  on  $]0, \delta[$  and

$$u'(r) = v(r) \quad \text{and} \quad v'(r) = -\frac{d-1}{r}v(r) - u(r)g(u(r)).$$

We now study the differentiability at 0. Using L'Hospital rule, we obtain

$$u''(0) = v'(0) = \lim_{r \rightarrow 0} \frac{v(r)}{r} = -\frac{ag(a)}{d}.$$

On the other hand,

$$\begin{aligned} \lim_{r \rightarrow 0} v'(r) &= \lim_{r \rightarrow 0} u''(r) \\ &= -\lim_{r \rightarrow 0} \left[ (d-1) \frac{v(r)}{r} + u(r)g(u(r)) \right] \\ &= -\left[ (d-1) \frac{ag(a)}{d} + ag(u(a)) \right] \\ &= -\frac{ag(a)}{d}. \end{aligned}$$

Hence,  $u$  is  $\mathcal{C}^2$  at 0. It suffices then to prove that  $u > 0$  on  $(0, +\infty)$  to guarantee the existence and uniqueness on  $(0, +\infty)$ . We suppose by contrast that  $u(\zeta) = 0$  for some  $\zeta > 0$ . The evaluation of the energy  $E$  gives

$$E(\zeta) = \frac{1}{2}u'^2(\zeta) < E(0) = F(a) < 0$$

because of the fact  $1 < a < p$ . Which leads to a contradiction.

Let  $u$  be a compactly supported solution of problem (2) already with  $a > 1$  and let  $R = \inf\{r \in (0, \infty), u(s) = 0, \forall s \geq r\}$ . Henceforth,  $u$  is a solution of the problem

$$\begin{cases} \Delta u + u - |u|^{-2\theta}u = 0 & \text{in } B(0, R), \\ u = 0 & \text{on } \partial B(0, R). \end{cases} \quad (4)$$

Recall that it is well known that for  $R < \sqrt{\lambda_1(B(0, 1))}$  the first eigenvalue of  $-\Delta$  on the unit ball, problem (4) has no positive solution. See [7], [8] and the references therein. Consequently, we will assume for the rest of this part that  $R \geq \sqrt{\lambda_1(B(0, 1))}$  and consider the radial expression of (4),

$$\begin{cases} u'' + \frac{d-1}{r}u' + u - |u|^{-2\theta}u = 0 & , \quad r \in (0, +\infty), \\ u(R) = 0. \end{cases} \quad (5)$$

We will discuss the behavior of the solution  $u$  relatively to the values  $u'(R)$ . Two situations can occur. First,  $u'(R) < 0$ . It results that  $u'(r) < 0$  on a small interval  $(R - \varepsilon, R + \varepsilon)$ . Therefore,  $u(r) < 0$  on  $(R, R + \varepsilon)$  which contradicts the definition of  $R$ . Next, for  $u'(R) = 0$ , we get  $E(R) = 0$

Denote for the rest of the paper  $\rho_a$  the first zero of the solution  $u$  of problem (2) for  $a > p$ . We have

**Lemma 2.3** *For all  $a > p$ ,  $\rho_a < \infty$ .*

*Proof.* Suppose  $u$  a solution of problem (2) with  $a > p$  and  $\rho_a = \infty$ . The solution  $u$  starts as decreasing from  $a = u(0)$ . Suppose that it remains decreasing on its whole domain  $(0, +\infty)$ . Thus it has a limit  $L$  as  $r \rightarrow +\infty$ . Thus  $L = 0$  or  $L = 1$ . For  $L = 0$  and  $r$  large enough, we obtain  $u(r) = A \cos(r) + B \sin(r)$  which is contradictory. The case where  $L = 1$  is analogous. Consequently  $\rho_a < +\infty$ .

We now study the behavior of the solution  $u$  on the whole domain  $(0, +\infty)$ . Denote  $r_0$  the first critical point of the solution  $u$  of problem (2) with  $a > p$ . There are four possible situations. The case  $u(r_0) > 1$  with equation (2) implies that

$$0 = \int_0^{r_0} (s^{d-1}u'(s))' ds = - \int_0^{r_0} s^{d-1}u(s)g(u(s))ds < 0$$

which is impossible. Next, for  $u(r_0) = 1$ , the solution  $u$  will be a solution of the problem

$$\begin{cases} u'' + \frac{d-1}{r}u' + u - |u|^{-2\theta}u = 0 & , \quad r \in (r_0, +\infty), \\ u(r_0) = 1 & , \quad u'(r_0) = 0. \end{cases} \quad (6)$$

Therefore,  $u \equiv 1$ , for any  $r \geq r_0$ , which is contradictory by the same argument as above. We now assume that  $0 < u(r_0) < 1$ . Implying Theorem 1.2 in [4], we observe that  $u$  is oscillating around 0 with no zeros. Which is contradictory. Now we examine the last case  $u(r_0) = 0$ . In this case, we obtain  $u(r_0) = u'(r_0) = 0$  and  $\rho_a > \sqrt{\lambda_1(B(0, 1))}$ .

### 3 Proof of Theorem 1.1.

We recall firstly that some situations which are somehow more general are developed in [4]. The proof developed here is inspired from there. Let  $a \in (0, 1)$  and  $u$  be the solution of problem (2). It holds that  $u''(r) > 0$  on a small interval  $(0, \varepsilon)$  for  $\varepsilon$  small enough positive. Consequently,  $u'$  is strictly increasing on  $(0, \varepsilon)$ . Which yields that  $u'(r) > 0$  on  $(0, \varepsilon)$ . Thus  $u$  is strictly increasing on  $(0, \varepsilon)$  for  $\varepsilon$  small enough positive. So that,  $u(r) > a$  on  $(0, \varepsilon)$ . We will prove that the value  $a$  is taken only for  $r = 0$ . Indeed, suppose not, and let  $\zeta > 0$  be the first point satisfying  $u(\zeta) = a$ . The evaluation of the energy  $E(r)$  at 0 and  $\zeta$  yields that

$$E(0) = F(a) > E(\zeta) = \frac{1}{2}u'^2(\zeta) + F(a)$$

which is contradictory. So, the solution  $u$  starts increasing with origin point  $u(0) = a$  and did not reach it otherwise. We next prove that it can not continue to increase on its whole domain  $(0, +\infty)$ . Suppose contrarily that it is increasing on  $(0, +\infty)$  and denote  $L$  its limit as  $r \rightarrow +\infty$ . Of course, such a limit can not be infinite because of the energy of the solution. Next, the finite limit is a zero of the function  $f(s)$ . Therefore,  $L = 1$ . But, this yields  $u''(r) > 0$  as  $r \rightarrow +\infty$  (Recall that  $f(s) < 0$  on  $(0, 1)$ ). In the other hand, equation (2) guaranties that  $\frac{u''(r)}{u'(r)} \sim -\frac{d-1}{r} < 0$  as  $r \rightarrow +\infty$  which means that  $u''(r) < 0$  as  $r \rightarrow +\infty$  leading to a contradiction. We therefore conclude that  $u$  is oscillatory. Let  $t_1$  be the first point in  $(0, +\infty)$  such that  $u'(t_1) = 0$ . It holds that  $u(t_1) > 1$ . If not, by multiplying equation (2) by  $r^{d-1}$  and integrating from 0 to  $t_1$  we obtain  $0 = -\int_0^{t_1} r^{d-1} f(u(r)) > 0$  which is contradictory. Thus,  $u$  crosses the line  $y = 1$  once in  $(0, t_1)$  leading to a unique point  $r_1 \in (0, t_1)$  such that  $u(r_1) = 1$ . Next, using similar techniques, we prove that  $u$  can not remain greater than 1 in the rest of its domain. (Consider the same equation on  $(t_1, +\infty)$  with initial data  $u(t_1)$  and  $u'(t_1)$ ). Consequently we prove that there exists unique sequences  $(t_k)_k$  and  $(r_k)_k$  such that

$$r_k < t_k < r_{k+1}, \quad u(r_k) = 1, \quad u'(\zeta_k) = 0, \quad k \geq 1. \quad (7)$$

Next, observing that  $E$  is decreasing as a function of  $r$ , we deduce that the sequence of maxima  $(u(t_k))_k$  goes to 1 and therefore  $u$ .

## 4 Proof of Theorem 1.2.

The proof is based on a series of preliminary results. We recall first that it suffices to study the case  $a > 0$  due to the parity properties of the function  $g$  and/or  $f$ .

**Lemma 4.1** *For  $a > 1$ , the solution  $u$  satisfies  $(u(r) < a, \forall r > 0)$ .*

*Proof.* From equation (2), we obtain  $du''(0) = -ag(a) < 0$ . Consequently,  $u''(r) < 0$  for  $r \in (0, \varepsilon)$  for some  $\varepsilon > 0$  small enough. Thus,  $u'$  is decreasing strictly on  $(0, \varepsilon)$  and then,  $u'(r) < 0$  for  $r \in (0, \varepsilon)$ . Therefore,  $u$  is decreasing strictly on  $(0, \varepsilon)$  and then,  $u(r) < a$  for  $r \in (0, \varepsilon)$ . Let next  $\zeta > 0$  be the first point such that  $u(\zeta) = a$ , if possible. Using the energy  $E$  we obtain  $E(\zeta) < E(r) < E(0)$ , for all  $r \in (0, \zeta)$ . Whenever  $u'(\zeta) = 0$ , we obtain  $E(0) < E(0)$  which is impossible. So  $u'(\zeta) \neq 0$ , which implies that  $\frac{1}{2}u'^2(\zeta) + E(0) < E(0)$  which is also impossible. As a conclusion, there is no positive points for which the solution  $u$  reaches  $a$  again.

**Lemma 4.2** *For  $a > 1$ , the solution  $u$  is not strictly decreasing on  $(0, +\infty)$ .*

*Proof.* Assume contrarily that  $u$  is strictly decreasing on  $(0, +\infty)$ . Thus, it has a limit  $L$  as  $r \rightarrow +\infty$ . Two cases are possible,  $L = 0$  or  $L = 1$ . We will examine them one by one.

*case 1.*  $L = 0$ . Consider the dynamical system in the phase plane defined for  $r \in (0, +\infty)$  by

$$\begin{cases} v = u', \\ v' = -\frac{d-1}{r}v + u - |u|^{-2\theta}u, \\ u(0) = a \end{cases}, \quad v(0) = 0 \quad (8)$$

A careful study for  $r \rightarrow +\infty$ , yields the estimation  $u \sim A \cos(r) + B \sin(r)$  for  $r$  large enough, which is contradictory.

*case 2.*  $L = 1$ . Using equation (2) or (8), we obtain for  $r$  large enough,  $2^{d-1}v(2r) - v(r) \sim \frac{2^d - 1}{d}r$  which leads to a contradiction.

**Lemma 4.3** *Let for  $a > 1$ ,  $r_1(a)$  be the first critical point of the solution  $u$  of (2) in  $(0, +\infty)$ . Then  $u(r_1) < 1$ .*

*Proof.* Suppose not, i.e,  $u(r_1) = 1$  or  $u(r_1) > 1$ . When  $u(r_1) > 1$ , we obtain  $u''(r_1) = -u(r_1)g(u(r_1)) < 0$ . Hence,  $u'$  is decreasing on  $(r_1 - \varepsilon, r_1 + \varepsilon)$  for some  $\varepsilon$  small enough. Thus,  $u$  is increasing near  $r_1$  at the left and decreasing near  $r_1$  at the right, which is contradictory. When  $u(r_1) = 1$ , then  $u$  is a solution of the problem  $u''(r) + \frac{d-1}{r}u'(r) + ug(u) = 0$  on  $(r_1, +\infty)$  with the initial condition  $u(r_1) = 1$  and  $u'(r_1) = 0$ . Consequently,  $u \equiv 1$  which is contradictory.

**Lemma 4.4** *Let  $a > 1$  and  $r_1(a)$  be the first critical point of the solution  $u$  in  $(0, +\infty)$ . Then*

- a.** *for  $r_1(a) \in ]0, 1[$ , the solution  $u$  of (2) oscillates around 1, with limit 1, and thus has a finite number of zeros.*
- b.** *for  $r_1(a) \in ]-1, 0[$ , the solution  $u$  of (2) oscillates around -1, with limit -1, and thus has a finite number of zeros.*

*Proof.* In the situation **a.**  $u$  is a solution of the problem

$$\begin{cases} u'' + \frac{d-1}{r}u' + u - |u|^{-2\theta}u = 0 & , \quad r \in (r_1(a), +\infty), \\ u(r_1(a)) \in ]0, 1[ & , \quad u'(r_1(a)) = 0. \end{cases} \quad (9)$$

Hence, by applying Theorem 1.1, the solution oscillates around 1, with limit 1 and thus, it has a finite number of zeros. In the situation **b.**  $u$  is a solution of the problem

$$\begin{cases} u'' + \frac{d-1}{r}u' + u - |u|^{-2\theta}u = 0 & , \quad r \in (r_1(a), +\infty), \\ u(r_1(a)) \in ]-1, 0[ & , \quad u'(r_1(a)) = 0. \end{cases} \quad (10)$$

Hence, for the same reasons, it oscillates around -1, with limit -1 and thus with a finite number of zeros.

**Lemma 4.5** *Let  $a > 1$  and  $u$  the solution of (2) in  $(0, +\infty)$ . The following situation can not occur. There exists sequences  $(r_k)$ ,  $(t_k)$ ,  $(z_k)$  and  $(\zeta_k)$  satisfying*

- i.**  $t_{2k-1} < z_{2k-1} < \zeta_{2k-1} < z_{2k} < t_{2k} < r_{2k} < \zeta_{2k} < r_{2k+1}$ ,  $\forall k$ .
- ii.**  $u(r_k) = -u(z_k) = 1$ ,  $u(t_k) = u'(\zeta_k) = 0$ ,  $\forall k$ .
- iii.**  $u$  is increasing strictly on  $(\zeta_{2k-1}, \zeta_{2k})$  and decreasing strictly on  $(\zeta_{2k}, \zeta_{2k+1})$ ,  $\forall k$ .

*Proof.* Suppose by contrast that the situation occurs. Using the functional energy  $E(r)$ , it is straightforward that  $|u(\zeta_k)| \downarrow 1$ . Observe next that for  $r$  large enough and  $k \in \mathbb{N}$  unique such that  $\zeta_{2k} \leq r < \zeta_{2k+1}$  or  $\zeta_{2k+1} \leq r < \zeta_{2k+2}$ , we have  $E(\zeta_{2k}) \leq E(r) < E(\zeta_{2k+1})$  or  $E(\zeta_{2k+1}) \leq E(r) < E(\zeta_{2k+2})$  which means that  $\lim_{r \rightarrow +\infty} E(r) = \frac{-\theta}{2(1-\theta)}$ . In particular we get  $\lim_{k \rightarrow +\infty} E(t_k) = \frac{-\theta}{2(1-\theta)}$ , which means that  $\lim_{k \rightarrow +\infty} u'^2(t_k) = \frac{-\theta}{1-\theta} < 0$  which is a contradiction.

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